Published by Institute of Physics Publishing for SISSA
Received: *July 3, 2006*
Revised: *October 5, 2006*
Accepted: *October 12, 2006*
Published: *November 7, 2006*

On the thermodynamic geometry of BTZ black holes

Tapobrata Sarkar, Gautam Sengupta and Bhupendra Nath Tiwari

Department of Physics, Indian Institute of Technology,
Kanpur 208016, India
E-mail: tapo@iitk.ac.in, sengupta@iitk.ac.in, bntiwari@iitk.ac.in

Abstract: We investigate the Ruppeiner geometry of the thermodynamic state space of a general class of BTZ black holes. It is shown that the thermodynamic geometry is flat for both the rotating BTZ and the BTZ Chern Simons black holes in the canonical ensemble. We further investigate the the contribution of thermal fluctuations to the canonical entropy of the BTZ Chern Simons black holes and show that the leading logartithmic correction due to Carlip is reproduced. We establish that the inclusion of thermal fluctuations induces a non zero scalar curvature to the thermodynamic geometry.

Keywords: Models of Quantum Gravity, Black Holes.

JHEP11(2006)015

Contents

1. Introduction **1**

2. Thermodynamic geometry of BTZ black holes **8**
- 2.1 Rotating BTZ black holes 8
- 2.2 Inclusion of thermal fluctuations 10

3. BTZ black holes with the Chern-Simons term **12**
- 3.1 BTZ-CS black holes with small fluctuations 14

4. Discussions and conclusions **16**

1. Introduction

Over the last few decades, black hole thermodynamics has been one of the most intense topics of research in theoretical physics (for a comprehensive review, see [1]). It is by now well known that black holes are thermodynamic systems, which posess a macroscopic entropy, and a characteristic Hawking temperature, related to the surface gravity on the event horizon. Indeed, these quantities satisfy the first three laws of thermodynamics although the third law in the Nernst form do not seem to apply to black holes. The macroscopic entropy of black holes is a function of their mass M, charge Q and angular momentum J describing a thermodynamic macrostate and is given as $S_{\rm macro} = S_{\rm macro}(M, J, Q)$. The entropy follows the Bekenstein-Hawking area law in standard Einstein gravity coupled to gauge fields such that $S_{\rm macro} = \frac{1}{4}A_h$ where A_h is the area of the event horizon. However an understanding of the microscopic statistical origin of black hole entropy has been an outstanding theoretical issue. Although considerable progress has been made in the recent past in matching microscopic state counting with the macroscopic entropy for extremal BPS black holes in string theory, a clear comprehension of the statistical microstates of black holes is still elusive. Recent interest in this area has been focussed on subleading corrections to the entropy from higher derivative terms in the low energy string theory effective action [2]. These corrections cause deviations from the area law and for certain classes of supersymmetric extremal black holes leads to exact maching upto subleading corrections with microscopic state counting in the associated confomal field theory.

It is well known that equilibrium thermodynamic systems posess interesting geometrical features [3]. An inner product on the equilibrium thermodynamic state space in the energy representation was provided by Weinhold [4] as the Hessian matrix of the internal energy with respect to the extensive thermodynamic variables as,

$$h_{ij} = \partial_i \partial_j U(S, N^a)$$

where U is the internal energy, S is the entropy and N^a stands for the other extensive variables and (i, j) runs over all the extensive variables. However there was no physical interpretation asociated with this metric structure. The Weinhold inner product was later formulated in the entropy representation by Ruppeiner [5] into a Riemannian metric in the thermodynamic state space. The Ruppeiner geometry was however physicaly meaningful in the context of equlibrium thermodynamic fluctuations of the system. The invariant line element between any two equlibrium states was related to the probability distribution of the fluctuations between them. The curvature scalar obtained from this geometry signified interactions and was proportional to the correlation volume which diverges at the critical points of phase transitions.[1]

The *Ruppeiner metric* on the thermodynamic state space, is defined as the Hessian of the entropy with respect to the extensive variables and is given by

$$g_{ij} = -\partial_i \partial_j S(U, N^a) \tag{1.1}$$

in the notation introduced earlier. The negative sign is necessary as entropy is a maximum for an equilibrium thermodynamic state in the entropy representation. It is to be noted that here the volume V is held fixed to provide a physical scale. The Ruppeiner metric is conformaly related to the Weinhold metric with the inverse temperature as the conformal factor. Assuming all the extensive variables to be labeled by x_i it is straightforward to show that the probability distribution of fluctuations $W(x)$ between two equilibrium states is given in the Gaussian approximation as

$$W(x) = A \;\; exp[-\frac{1}{2} g_{ij}(x) dx^i dx^j]. \tag{1.2}$$

Whilst the inverse metric may be shown to be the second moment of fluctuations or the pair correlation functions and given as $g^{ij} = < x^i x^j >$.

For a standard two dimensional thermodynamic state space defined by the extensive variables (x^1, x^2) the application of these geometric notions to conventional thermodyamic systems suggest that the scalar curvature indicates interactions. It may be shown that $R \sim \kappa_2 \xi^d$ where ξ is the correlation length, d is the dimensionality and κ_2 is a dimensionless constant of order one. A few simple manipulations illustrate that the thermodynamic curvature is inversely proportional to the singular part of the free energy associated with long range correlations which diverge at the critical point of phase transitions. So the thermodynamic curvature may be expressed as $R = -\frac{\kappa k_B}{\phi}$ where $\phi = -\frac{\kappa_1 k_B}{\xi^d}$ and its divergence signifies a phase transition. The Ruppeiner formalism has been applied to different condensed matter systems and is completely consistent with the scaling and hyperscaling relations involving critical phenomena and have reproduced the corresponding critical indices.

Although, isolated asymptoticaly flat black holes do not follow the usual precepts of extensive thermodynamic systems it is possible to consider the black hole entropy as an

[1]This is true only for a two dimensional thermodynamic geometry. For higher dimensional geometries all the indepedent nonzero components should be significant in the determination of interactions.

extensive thermodynamic quantity provided the black hole is a part of a larger system with which it is in thermal equilibrium. From this perspective the geometric notions of thermodynamics may be applied to investigate the nature of the black hole entropy. In particular the investigation of the covariant thermodynamic geometry of Ruppeiner for black holes have elucidated interesting aspects of black hole phase transitions and relations to moduli spaces. This was first explored in the context of extremal charged BPS black hole configurations of $N \geq 2$ supergravity in $D = 4$ which arises as the low energy effective theory from type II string compactifications [6]. Since then, several authors have attempted to understand this connection [7−10] both for supersymmetric as well as non-supersymmetric black holes. and five dimensional rotating black rings.

The charged extremal black holes of N=2 supergravity in D=4, interacting with n vector multiplets are described by a Reissner-Nordstorm metric and are BPS states preserving a fraction of the full N=2 supersymmetry. The n vector multiplets involve $(n + 1)$ gauge fields and n complex scalar fields ϕ^a. The black hole solutions are hence characterized by electric and magnetic charges q_J and p^I arising from usual flux integrals of the field strengths and their Poincare duals. The scalar fields on the other hand serves as moduli which parametrizes the compact internal space. The extremal charged black hole solutions are BPS solitons which interpolate between asymptotic infinity and the near horizon geometry. The spherical symmetry determines this interpolation to be a radial evolution of the scalar moduli which thus encodes the consequent changes in the underlying internal compact manifold. At asymptotic infinity one has flat Minkowski space with the scalar moduli tending to certain arbitrary values with the ADM mass given by $M(p, q, \phi^a_\infty) =\mid Z_\infty \mid$ where Z_∞ is the complex central extension of the supersymmetry algebra. The near horizon geometry is described by a $AdS_2 \times S^2$ charged Bertotti-Robinson metric. The area of the horzion and hence the macroscopic entropy is given as $S_{\text{macro}} = \frac{A}{4} = \pi \mid Z \mid^2_{\text{hor}}$. Thus *a priori* the entropy seems to depend on both the charges as well as the values of the scalar moduli which maybe changed continuously. This is incompatible with the interpretation of the entropy in terms of some microscopic state counting in an underlying statistical system.

However the radial variation of the moduli may be described by a damped geodesic equation which flows to an attractive fixed point at the horizon determined by the charges. The equations governing the scalar moduli in terms of the charges at the horzion are thus refered to as *attractor equations*. This ensures that the entropy is a function of the charges only and independent of the values of the moduli. The geodesic flow of the *attractor mechanism* involves a black hole *effective potential* $V(p, q, \phi^a)$ which is a symplectic invariant of the N=2 special geomtery. To each critical point ϕ^a_h of the effective potential $V(q, p, \phi)$ such that $(\frac{\partial V}{\partial \phi^a})_h = 0$ on the moduli space $\mathcal{M}_\phi$, a supersymmetric Bertotti-Robinson vaccuum state may be associated. So the attractor flow essentialy interpolates between the asymptoticaly flat vaccuum and the near horizon Bertotti-Robinson metric with the moduli starting from their asymptotic values ϕ^a_∞ and flowing to the attractor values ϕ^a_h at the horizon. The critical points of the potential are also the critical points of the associated central charges where it is a minimum. The entropy is then given as $S_{\text{macro}} = \frac{A}{4} = \pi V(p, q, \phi^a_h)$.

The thermodynamic state space for these black holes away from the critical points would now involve also the moduli fields ϕ^a_∞ at asymptotic infinity apart from the electric and the magnetic charges [11]. This leads to a curved theromdynamic state space $R^{2n} \otimes \mathcal{M}_\phi$. The thermodynamic variables conjugate to the moduli ϕ^a_∞ are negative of the scalar charges Σ which serves as a chemical potential. So the first law is modified to [11]

$$dM = TdS + \psi^A dQ_A + \chi^A dP_A - \Sigma^a d\phi^a_\infty$$

where (ψ^A, χ^A) were the electrostatic and magnetostatic potentials respectively.

It is interesting to ask how $V(p, q, \phi^a)$ defined on the moduli space of scalars $\mathcal{M}_\phi$ behaves as a function of the moduli. A generalization of the Ruppeiner metric to include the moduli as extensive thermodynamic variables was proposed in [6] as the Hessian of the effective potential;

$$g_{ij} = -\nabla_i \nabla_j V(p, q, \phi^a_\infty)$$

where the covariant derivatives are with respect to the moduli space metric. Notice that $\pi V(p, q, \phi)$ reduces to the entropy at the critical point. The positivity of this Riemannian metric in the extended thermodynamic state space provides a convexity condition leading to the critical points being an extremum of the potential. This metric also turns out to be proportional to the moduli space metric.

Although a thermodynamic interpretation of the effective potential away from the attractor fixed points is not clear we may explore the properties of the moduli space through the associated extension of the thermodynamic geometry in terms of the Hessian of the *effective potential*. In particular the curvature and curvature invariants should provide information about the singularities of the moduli space. Furthermore such a geometric analysis may provide useful insight into the full structure of the BPS black holes away from the attractor fixed point at the horizon and the attractor flows as certain renormalization group flows. Application of these geometric ideas to non extremal black holes are expected to be particularly significant. Additionaly such an analysis may elucidate the role of thermodynamics away from the attractor fixed points.

The low energy effective action of N=2 supergravity interacting with n vector multiplets following from type II string compactifications involve higher curvature terms in an α' expansion. These higher derivative terms modify the Bekenstein Hawking area law and introduces subleading corrections to the entropy of black holes. The effect of the higher derivative terms may be analysed in Wald's framework for generaly covariant higher derivative gravity. The entropy of extremal black holes in this analysis follows from the surface integral of a Noether charge density over the black hole horizon. The resulting entropy matches the conventional one at the two derivative level.

The higher derivative terms in the sueprgravity effective action are encoded in the generalized prepotential $F(Y, \Upsilon)$ which is a holomorphic function, homogeneous of degree two in the rescaled complex scalar fields Y and the anti-self dual part of graviphoton field Υ. The *attractor mechanism* continues to hold in the presence of the higher derivative terms. An application of Walds analysis to a suitably chosen prepotential yields the macroscopic entropy as

$$S_{\text{macro}} = \pi(\mid Z \mid^2 - 4Im(\Upsilon F_\Upsilon)) \mid_{\text{hor}}$$

JHEP11(2006)015

where $F_\Upsilon = \frac{\partial F}{\partial \Upsilon}$ and

$$| Z |^2 = (p^I F_I - q_I Y^I)$$

and $F_I = \frac{\partial F}{\partial Y^I}$. The attractor equations may be solved to determine the scalar fields in terms of the charges at the horizon and this ensures that the macroscopic entropy is a function of the charges alone.

The BPS black hole solutions in N=2 supergravity fall in two distinct classes namely the *large black holes* which have a non vanishing area at the two derivative level and posess dyonic charges and the *small black holes* which have a vanishing area and carry electric charges only. The *large black holes* may be described in terms of wrapped branes on non-trivial cycles of the compact internal manifold . The microscopic entropy is then determined in terms of the microstate counting through the Cardy formula in the underlying two dimensonal CFT associated with the brane system. The microscopic entropy computed from this is in precise agreement upto the subleading terms with the macroscopic entropy following from the Wald formula. It must be emphasized here that the precise matching of the Wald and the Cardy formula involves the extremal BTZ black hole. This is because the near horizon geometry of $AdS_2 \times S^2$ has a local AdS_3 structure which is asymptotic to the BTZ black hole.

The case for the *small black holes* turn out to be more complicated. The microscopic entropy matches the macroscopic one at leading orders but fails at subleading orders. S-duality requires the addition of non holomorphic terms to the generalized pre-potential. This leads to a required S-duality invariant macroscopic entropy with subleading corrections but the coefficients still show a mismatch. A reformulation of the problem in terms of the black hole free energy related to the logarithm of the black hole partition function indicates that the corresponding ensemble must be a mixed ensemble. This proposal admits a direct relation between the black hole entropy and the corresponding topological string partition function. However it seems to have several problems regarding duality invariance and a non perturbative completion.

It was observed that the attractor equations and the macroscopic entropy follow directly from a variational principle applied to a generic class of *entropy functions* $\Sigma(p, q, Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$ of the charges, rescaled scalar moduli fields and the relevant part of the graviphoton field. The condition for the extremum of Σ leads to the attractor equations and at the attractor point which is reached at the horizon $S_{\text{macro}} = \pi \Sigma \mid_h$. The non holomorphic contributions may also be incorporated in the variational framework with a modification of the *entropy function* Σ. The proposal regarding the black hole partition function may be now recast in the variational language of the *entropy function.* Although this obviates most of the earlier problems the coefficients for the subleading terms still shows a mismatch with the microscopic state counting.

It would be interesting to study the effect of the higher derivative corrections to the thermodynamic geometry of these extremal black holes. Small black holes in string theory are particularly interesting in this connection as their horizon area and entropy vanishes at the two derivative level. So the thermodynamic geometry arises only from the macroscopic entropy following from the higher derivative corrections to the area . Although at near

JHEP11(2006)015

extreme the thermodynamics seems to break down geometric quantities are still well defined as is the entropy. It is particularly important to study and interpret the sensitivity of the scalar curvature to the higher derivative corrections in the macroscopic entropy. This may provide useful insight into the relation of thermodynamic breakdown with extremality.

It is also important to study the full structure of these interpolating solutions of the higher derivative theory away from the horizon. This would naturaly involve the *entropy function* Σ away from the attractor point. Although a thermodynamic interpretation of the *entropy function* Σ away from the attractor fixed point is not clear, a generalization of the Ruppeiner metric may be proposed as the Hessian of the *entropy function* with respect to all the extensive thermodynamic variables including the moduli fields as

$$g_{ij} = -\nabla_i \nabla_j \Sigma((p, q, Y, \bar{Y}, \Upsilon, \bar{\Upsilon}),$$

where the covariant derivatives are with respect to the moduli space metric. Following [6] it seems plausible that this metric would also be proportional to the moduli space metric. Study of the curvature and the curvature invariants of this generalized thermodynamic geometry is hence expected to provide important insight regarding the singularities in moduli space and their significance. This may also lead to a viable thermodynamic interpretation away from the attractor fixed points. In particular it may be possible to understand the attractor mechanism as some renormalization group flows in the space of thermodynamic geometries.

The simplest black hole system for which it is possible to analyze the thermodynamic geometry is the three dimensional rotating BTZ black hole which as mentioned earlier plays a particularly significant role in the attractor mechanism. Here, one may construct the Ruppeiner metric in terms of the black hole mass and its angular momentum (the non-rotating BTZ black hole is trivial), and it turns out that this metric is flat[2] [10].[3]

Recently Kraus and Larsen [12] and Solodukhin [13] have shown how various properties of BTZ black holes are affected by the addition of the gravitational Chern-Simons term to the three dimensional Einstein-Hilbert action. In particular, they show that the black hole entropy is modified by the presence of this term and obtain an explicit expression for this modified entropy. In this context, it is imperative to re-examine the Ruppeiner geometry of the BTZ black hole, in the presence of the Chern-Simons term.

A black hole in equilibrium with the thermal Hawking radiation at a fixed Hawking temperature is conventionaly described by a canonical ensemble. Although it now appears in the light of recent observations that a mixed ensemble may be more appropriate we stick to the conventional approach in this article. The thermodynamic geometry of the black hole entropy function has hence been determined with reference to the canonical ensemble. However its well known that thermal fluctuations in the canonical ensemble generates logarithmic corrections to the entropy [14]. These corrections vanish in the thermodynamic limit where the canonical and the microcanonical entropy are identical. For black holes such

[2]Ricci flatness in two dimensions implies a flat space.

[3]See ref. [9] and [10] for a classification of the nature of the Ruppeiner metrics for black holes in various dimensions.

JHEP11(2006)015

logarithmic corrections to the canonical entropy have been obtained in [15, 16]. It is a natural question as to whether the thermodynamic geometry of black holes are sensitive to these fluctuations. As we will show in the next few sections, thermal fluctuations indeed modify the Ruppeiner geometry of the BTZ black-holes with and without the Chern-Simons term.

In another interesting development, Sen and later Sahoo and Sen [17] have computed the BTZ black hole entropy in the presence of the Chern-Simons and higher derivative terms [18]. A variant of the *attractor mechanism* involving the use of the Sen entropy function $\mathcal{E}(p, q, u, v)$ was applied to an effective two-dimensional theory that results upon making the angular coordinate of the BTZ solution as a compact direction. Here (p, q) are the charges and (u, v) are the moduli parameters describing the solution. The attractor equations follow from an extremization of the entropy function and at the attractor fixed point $S_{\text{macro}} = \mathcal{E}$. It is indeed natural to examine the thermodynamic geometry of BTZ Chern Simons black holes with such higher derivative terms and investigate the effect of thermal fluctuations on this geometry. In particular the sensitivity of the scalar curvature to the higher derivative corrections may lead to useful insight into the thermodynamic behaviour.

A possible extension of the thermodynamic geometry may be proposed as the Hessian of the Sen entropy function as $\mathcal{E}$ as $g_{ij} = -\nabla_i \nabla_j \mathcal{E}(p, q, u, v)$ although the thermodynamic interpretation away from the attractor fixed point is not quite clear. Following [6] this should be proportional to the moduli space metric and may be used to study the moduli space geometry. It should also be possible to understand the attractor flows in the context of the thermodynamic geometries. The Sen entropy function $\mathcal{E}$ is particularly suited for this analysis as it assumes no supersymmetry and this considerably simplifies the computations. We should mention here that the BTZ black hole solution has an effective one dimensional moduli space which is geomtricaly trivial. However the geometrical analysis may be also applied to other black hole solutions described earlier which posess interesting moduli space geometries.

It is to be emphasized here that the thermal fluctuations in the canonical ensemble may be analysed through purely thermodynamic considerations. In contrast the α' corrections to the black hole entropy from higher derivative terms in the effective action may be analysed from a purely gravitational perspective through Walds analysis of a generaly covariant theory of higher curvature gravity [1]. Although the structures of the corrections are simmilar they may enter with opposing signs leading to a cancellation. In addition there should be quantum corrections following from purely quantum gravitational effects. We note that it is not meaningful to analyse corrections due to thermal fluctuations over and above corrections due to pure quantum gravity effects.

It is the above considerations, that we set out to explore in this paper in the context of the BTZ black hole which is the simplest black hole system to analyze. As indicated earlier the BTZ black hole is particularly significant for the matching of macrscopic and micrscopic entropies at subleading orders. Our main result is that the thermodynamic geometry is flat for the rotating BTZ black hole in the presence of the Chern-Simons and higher derivative terms. We show that inclusion of thermal fluctuations non-trivially modify the thermodynamic geometry of the BTZ black hole both with and without the Chern-Simons and the higher derivative corrections. As a by-product of our results, we show that the leading order

JHEP11(2006)015

correction to the canonical entropy of the BTZ black hole due to thermal fluctuations are reproduced in the presence of Chern-Simons terms also illustrating further the universality of these corrections.[4] Additionaly we have also explored an extension of the thermodynamic geometry based on the Hessian of the Sen entropy function with respect to the charges and the moduli as extensive variables in the spirit of [11]. It turns out the thermodynamic geometry of the BTZ black hole remains flat both with and without the Chern-Simons term away from the attractor point and the curvature is insensitive to the higher derivative corrections even at the attractor fixed point. We point out here that this is possibly due to the simple structure of the BTZ black hole and associated trivial one dimensional moduli space with no interesting geometry. We are currently exploring other extremal black hole solutions of N=2 supergravity which clearly shows the thermodynamic curvature to be sensitive to the higher derivative contributions and non holomorphic corrections [20].

The article is organized as follows. In section 2, we first review some known facts about the thermodynamic geometry for BTZ black holes, mainly to set the notations and conventions used in this paper, and then examine the thermodynamic geometry of BTZ black holes including small thermodynamic fluctuations. In section 3, we examine the Ruppeiner geometry of the BTZ black hole in the presence of the Chern-Simons term [13] and show that including small fluctuations in the analysis, the leading order correction to the entropy turns out to be the same as that of [16]. We then calculate the Ruppeiner curvature scalar and verify the bound on the Chern-Simons coupling, as predicted by Solodukhin. Section 4 contains some comments on higher derivative corrections to the BTZ black hole entropy, and discussions and directions for future investigations. Some of the calculations are unfortunately too long to reproduce here, and whereever necessary, we have used numerical techniques to highlight and illustrate our results.

2. Thermodynamic geometry of BTZ black holes

In this section, we study certain aspects of the thermodynamic (Ruppeiner) geometry of BTZ black holes. We will use the units $8G_N = \hbar = c = 1$ and start by reviewing the results for the rotating BTZ black hole and then examine the role of small thermal fluctuations.

2.1 Rotating BTZ black holes

The purpose of this subsection is mainly to set the notations and conventions that will be followed in the rest of the paper. We start with the BTZ metric

$$ds^2 = -N(r)dt^2 + \frac{1}{N(r)}dr^2 + r^2\left(N^\phi dt + d\phi\right)^2 \tag{2.1}$$

where N and N^ϕ are the (squared) lapse and shift functions defined by

$$N(r) = \frac{J^2}{4r^2} + \frac{r^2}{l^2} - M; \quad N^\phi = -\frac{J}{2r^2} \tag{2.2}$$

[4]See, for eg. [19] for some related works on BTZ black hole entropy, its logarithmic corrections, and logarithmic corrections to certain other classes of black hole geometries.

JHEP11(2006)015

with M and J being the mass and the angular momentum of the black hole, and l^2 represents the Cosmological constant term. The BTZ black hole has two horizons, located at

$$r_{\pm} = \sqrt{\frac{1}{2} M l^2 \left(1 \pm \Delta\right)} \tag{2.3}$$

where

$$\Delta = \sqrt{1 - \frac{J^2}{M^2 l^2}} \tag{2.4}$$

The mass and angular momentum of the black hole may be expressed in terms of $r_{\pm}$ of eq. (2.3) as;

$$M = \frac{r_+^2 + r_-^2}{l^2}; \qquad J = \frac{2 r_+ r_-}{l} \tag{2.5}$$

The BH entropy of the ordinary BTZ black hole is given by

$$S = 4\pi r_+ \tag{2.6}$$

The Ruppeiner metric is two dimensional, and is a function of the black hole mass M and angular momentum J. Explicitly, the metric is given by

$$g_{ij} = -\begin{pmatrix} \frac{\partial^2 S}{\partial J^2} & \frac{\partial^2 S}{\partial J \partial M} \\ \frac{\partial^2 S}{\partial J \partial M} & \frac{\partial^2 S}{\partial M^2} \end{pmatrix} \tag{2.7}$$

with $i, j \equiv J, M$.

We will use this general form of the Ruppeiner metric throughout this paper. A simple calculation shows that the Christoffel symbols are given by [5]

$$\begin{aligned} \Gamma_{JJJ} &= -\frac{1}{2}\frac{\partial^3 S}{\partial J^3} \quad \Gamma_{MMM} = -\frac{1}{2}\frac{\partial^3 S}{\partial M^3} \quad \Gamma_{JJM} = -\frac{1}{2}\frac{\partial^3 S}{\partial M \partial J^2} \\ \Gamma_{JMJ} &= -\frac{1}{2}\frac{\partial^3 S}{\partial M \partial J^2} \quad \Gamma_{JMM} = -\frac{1}{2}\frac{\partial^3 S}{\partial J \partial M^2} \quad \Gamma_{MMJ} = -\frac{1}{2}\frac{\partial^3 S}{\partial M^2 \partial J} \end{aligned} \tag{2.8}$$

with the symmetries relating the other components. The only non-vanishing component of the Riemann-Christoffel curvature tensor is $R_{JMJM} = N/D$, where

$$\begin{aligned} N = \; & \left(\frac{\partial^2 S}{\partial J^2}\right)\left[\left(\frac{\partial^3 S}{\partial M \partial J^2}\right)\left(\frac{\partial^3 S}{\partial M^3}\right) - \left(\frac{\partial^3 S}{\partial J \partial M^2}\right)^2\right] \\ & + \left(\frac{\partial^2 S}{\partial M^2}\right)\left[\left(\frac{\partial^3 S}{\partial J \partial M^2}\right)\left(\frac{\partial^3 S}{\partial J^3}\right) - \left(\frac{\partial^3 S}{\partial M \partial J^2}\right)^2\right] \\ & + \left(\frac{\partial^2 S}{\partial J \partial M}\right)\left[\left(\frac{\partial^3 S}{\partial M \partial J^2}\right)\left(\frac{\partial^3 S}{\partial J \partial M^2}\right) - \left(\frac{\partial^3 S}{\partial J^3}\right)\left(\frac{\partial^3 S}{\partial M^3}\right)\right] \end{aligned} \tag{2.9}$$

and

$$D = 4\left[\left(\frac{\partial^2 S}{\partial J^2}\right)\left(\frac{\partial^2 S}{\partial M^2}\right) - \left(\frac{\partial^2 S}{\partial J \partial M}\right)^2\right] \tag{2.10}$$

[5] Our notation is $\Gamma_{ijk} = g_{ij,k} + g_{ik,j} - g_{jk,i}$

JHEP11(2006)015

The Ricci scalar is

$$R = \frac{2}{\det g} R_{JMJM} \tag{2.11}$$

It is easy to compute the Ricci scalar by using

$$r_+ = \frac{1}{2}\left[\sqrt{l\,(Ml + J)} + \sqrt{l\,(Ml - J)}\right] \tag{2.12}$$

Using eq. (2.12) in eqs. (2.6), (2.11), (2.9) and (2.10), it can be easily shown that the Ricci scalar vanishes identically [10].

We might point out here that in [7], from considerations of the laws of black hole thermodynamics, the authors have argued that the internal energy of a charged or rotating black hole might not always be equal to its mass. Although we are not in full agreement with the arguments of [7], we have checked nevertheless that a modification of the internal energy of the rotating BTZ black hole in lines with [7] does not change the observation above.

2.2 Inclusion of thermal fluctuations

We will now discuss the Ruppeiner geometry of BTZ black holes including thermal fluctuations about the equilibrium. As is well known, any thermodynamical system, considered as a canonical ensemble has logarithmic and polynomial corrections to the entropy [14]. These considerations apply to black holes as well (considered as a canonical ensemble), and the specific forms of the logarithmic and polynomial corrections has been calculated for a wide class of black holes in [15]. It is to be noted that the applicability of this analysis presupposes that the canonical ensemble is thermodynamicaly stable. This requires a positive specific heat or correspondingly the Hessian of the entropy function must be negative definite.

The microcanonical entropy for any thermodynamical system, incorporating such corrections, is [14]

$$S = S_0 - \frac{1}{2}\ln\left(CT^2\right) \tag{2.13}$$

where S_0 is the entropy calculated in the canonical ensemble, and S is the corrected microcanonical entropy. C is the specific heat, and it is understood that appropriate factors of the Boltzmann's constant are included to make the logarithm dimensionless. The approximation is valid only in the regime where thermal fluctuations are much larger than quantum fluctuations. In [15], the BTZ black hole was analysed in this framework and eq. (2.13) reproduces the leading order correction to the entropy as obtained in [16]. It is then a natural question as to how the Ruppeiner geometry for the BTZ black hole is modified due to the thermal fluctuations in the canonical ensemble and this is what we will analyse in the rest of this section.

The Ruppeiner metric for the corrected entropy for the BTZ black hole of eq. (2.13) can be calculated using the equations (2.9), (2.10) and (2.11). Since the expressions involved are lengthy, we will set the cosmological constant $l = 1$. The Hawking temperature of the BTZ black hole is given by

$$T_H = \frac{1}{2\pi}\left[\frac{r_+^2 - r_-^2}{r_+}\right] \tag{2.14}$$

JHEP11(2006)015

which can be readily expressed in terms of the entropy of eq. (2.6) as

$$T_H = \frac{S}{8\pi^2} - \frac{8\pi^2 J^2}{S^3} \tag{2.15}$$

The specific heat is

$$C = \left(\frac{\partial M}{\partial T}\right)_J = \frac{S\left(S^4 - 64\pi^4 J^2\right)}{S^4 + 192\pi^4 J^2} \tag{2.16}$$

The specific heat is positive and this ensures that the stability of the corresponding canonical ensemble. Alternatively the Hessian of the internal energy (ADM mass) with respect to the extensive variables in the energy representation is given as

$$|| \frac{\partial^2 M}{\partial X_i \partial X_j} ||= \frac{1}{S^2 l^2} - \frac{64\pi^4 J^2}{S^6}$$

. This is positive provided $\frac{J}{S^2} < 1$ ensuring the thermodynamic stability of the corresponding BTZ black hole. It is to be noted that this condition also governs the situation away from extremality. Substituting the expressions of (2.15) and (2.16) in (2.13), we obtain the corrected entropy of the BTZ black hole, and the Ruppeiner metric for this entropy. The expression of the curvature scalar of this metric is far too complicated to present here, so we present the results numerically.

First, we consider the Ruppeiner metric with just the leading logarithmic correction of [16]. In this case, the analysis is simplified and (2.13) reduces to

$$S = S_0 - \frac{3}{2}\ln S_0 \tag{2.17}$$

Figure (1) shows the curvature scalar of the Ruppeiner metric, R, plotted against the angular momentum J for $M = 100$, where we have taken only the logarithmic correction of eq. (2.17) to the entropy into account. We have restricted to small values of J, so that we are far from extremality, i.e in the regime where these results are valid. Indeed, for near extremal BTZ black holes (i.e for very low temperatures), our analysis is not valid [15]. ¿From figure (1), we see that in this case, the curvature scalar is not positive definite, and indeed, by extending the values of J, it is seen that the curvature scalar goes to zero at J increases towards its extremal value. However, we must point out that our calculations that lead to this result can only be trusted when the black hole is far from extremality. Also note that even at zero angular momentum, there is a small but finite value of the curvature scalar. This indicates that even at zero angular momentum, the statistical system is interacting, once small fluctuations are included. This should be contrasted with the non-rotating BTZ black hole which is a non-interacting system even when small fluctuations are included. We have checked that increasing the value of M, the value of the curvature scalar becomes smaller, while preserving the shape of the graph.

Figure (2) shows the Ruppeiner curvature scalar plotted against the angular momentum, calculated using eq. (2.13). Interestingly, in this case, the Ruppeiner scalar is positive definite. Again, we have restricted ourselves to values of J small compared to M (i.e far from extremality) where our results can be trusted.

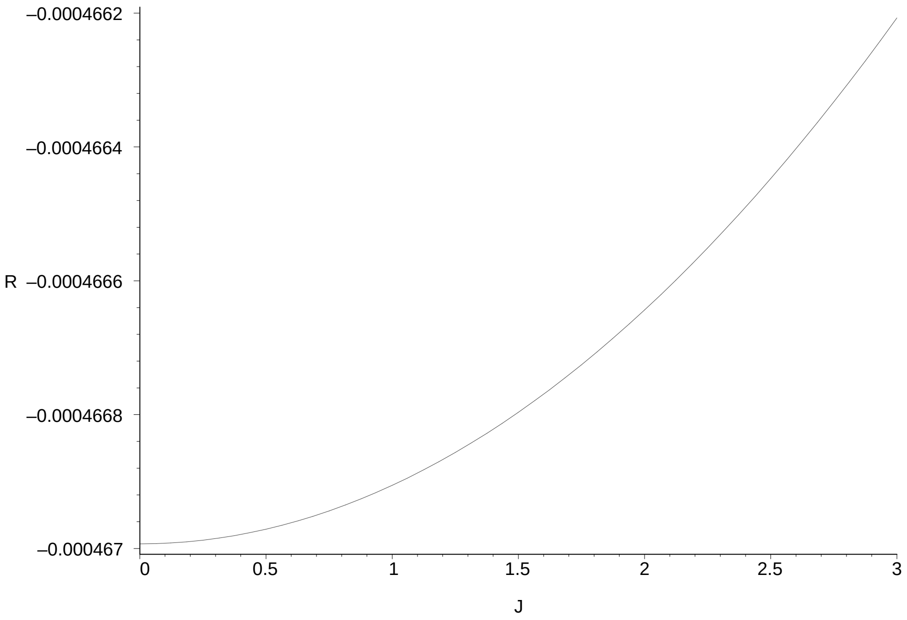

Figure 1: The Ricci scalar R of the Ruppeiner metric for the BTZ black hole, as a function of the angular momentum J, with only the logarithmic correction to the entropy (eq. 2.17) being taken into account. The mass M has been set to 100.

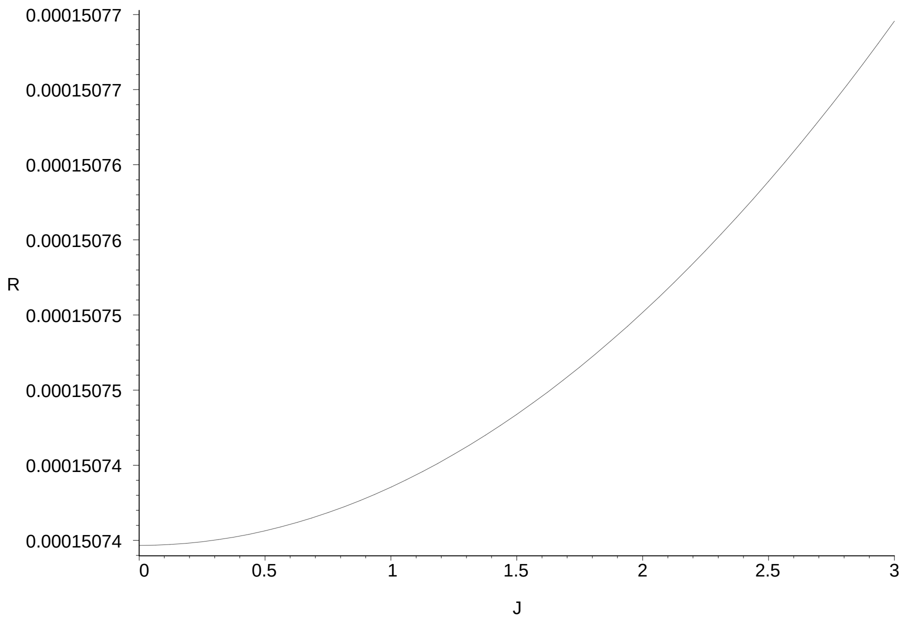

Figure 2: The Ricci scalar R of the Ruppeiner metric for the BTZ black hole, as a function of the angular momentum J, including small fluctuations (eq. 2.13). The mass M has been set to 100.

3. BTZ black holes with the Chern-Simons term

Recently, Kraus and Larsen [12] and Solodukhin [13] have studied gravitational anomalies for three-dimensional gravity in the presence of the Chern-Simons term. Indeed, the BTZ black hole is a bonafide solution to the gravitational action that included both the Einstein-

Hilbert and the Chern-Simons term. We will henceforth refer to the BTZ black hole with the Chern-Simons term as the BTZ-CS black hole.In [12, 13], the entropy of BTZ-CS black holes have been analysed, and these authors have derived an expression for the entropy, which differes from the entropy of the "usual" BTZ black hole, eq. (2.6). The modified entropy for the BTZ-CS black hole is

$$S = 4\pi \left(r_+ - \frac{K}{l} r_- \right) \tag{3.1}$$

where K is the Chern-Simons coupling. The extra term in eq. (3.1) as compared to eq. (2.6) is the contribution from the Chern-Simons term and has very interesting properties. In particular, [13] predicts a stability bound

$$|K| \leq l \tag{3.2}$$

on the Chern-Simons coupling, from physical considerations. In view of the above, it is natural to ask what type of Ruppeiner geometry is seen by the BTZ black hole in the presence of the Chern-Simons term and it is this issue that we address in this section.

It is important to remember here that the usual mass and angular momentum of the BTZ black hole is modified in the presence of the Chern-Simons term. This may be calculated by integrating the modified stress tensor of the theory using the Fefferman-Graham expansion of the BTZ metric and reads [13]

$$M = M_0 - \frac{K}{l^2} J_0; \qquad J = J_0 - K M_0 \tag{3.3}$$

where M_0 and J_0 are the the mass and angular momentum of the usual BTZ black hole of eq. (2.5). We have calculated the Ruppeiner metric for the BTZ black hole (with the thermodynamic coordinates now being M and J, rather than M_0 and J_0) in the presence of the Chern-Simons term, taking into account the modifications of the mass and angular momentum as in eq. (3.3).[6] Writing the entropy as

$$S = 2\pi \left[\sqrt{(1-K)(M+J)} + \sqrt{(1+K)(M-J)} \right] \tag{3.4}$$

it is easy to calculate the geometric quantities. The expressions leading to the calculation of the Ricci scalar are not important, and we simply point out that the curvature scalar for this geometry turns out to be zero, i.e, the Ruppeiner geometry of the BTZ-CS black hole is flat showing that it is a non interacting statistical system. This is the main result of this subsection. Additionaly we would like to state that we have explicitly studied the generalization of the Ruppeiner metric away from the attractor fixed point at the horizon as the Hessian of the Sen entropy function $\mathcal{E}(p, q, u, v)$ with respect to both the charges and the moduli fields. We have verified that the thermodynamic curvature is insensitive to the higher derivative corrections and remains flat both at the horizon and away from it. This is possibly due to the fact that the effective moduli space of BTZ black holes is one dimensional and posess no interesting geometric structure.

[6] We have set the cosmological constant $l = 1$ for simplicity.

JHEP11(2006)015

3.1 BTZ-CS black holes with small fluctuations

We will now discuss some thermodynamic properties of the BTZ-CS black holes, treating the system as a canonical ensemble. We allow for small thermal fluctuations of the system considered as a canonical ensemble, and study the thermodynamic geometry of the BTZ-CS black hole in lines with our treatment of the usual rotating BTZ black hole described earlier

As before, we would like to analyse the Ruppeiner metric for the BTZ-CS black hole, with the entropy now being given by eq. (3.1). Again, for ease of notation, we set the cosmological constant $l = 1$. We begin by expressing the outer and inner horizons of the BTZ-CS black hole as

$$r_+ = \frac{1}{2}\left[\sqrt{M_0 + J_0} + \sqrt{M_0 - J_0}\right]$$
$$r_- = \frac{1}{2}\left[\sqrt{M_0 + J_0} - \sqrt{M_0 - J_0}\right]. \tag{3.5}$$

In terms of the corrected mass and angular momentum of eq. (3.3), these expressions become

$$r_+ = \frac{1}{2}\left[\sqrt{\frac{M+J}{1-K}} + \sqrt{\frac{M-J}{1+K}}\right]$$
$$r_- = \frac{1}{2}\left[\sqrt{\frac{M+J}{1-K}} - \sqrt{\frac{M-J}{1+K}}\right] \tag{3.6}$$

The equation for the entropy, given by (3.1), can now be solved to obtain the mass M in terms of S and J, and gives

$$M = \frac{1}{2K^2}\left[\left(2KJ + \frac{S^2}{4\pi^2}\right) + \left[\left(2KJ + \frac{S^2}{4\pi^2}\right)^2 - 4K^2\left(\frac{S^4}{64\pi^4} + \frac{S^2KJ}{4\pi^2} + J^2\right)\right]^{\frac{1}{2}}\right] \tag{3.7}$$

The temperature of the BTZ-CS black hole, given by $\left(\frac{\partial M}{\partial S}\right)_J$ may be obtained from the expression for M, and is given by

$$T = \frac{SK^2\left[S^2\left(1-K^2\right) + 8JK\pi^2\left(1-K^2\right) + \left[S^2\left(1-K^2\right)\left(S^2 + 16\pi^2 JK\right)\right]^{\frac{1}{2}}\right]}{4\pi^2\left[S^2\left(1-K^2\right)\left(16\pi^2 KJ + S^2\right)\right]} \tag{3.8}$$

The specific heat may be calculated from the expression

$$C = \left(\frac{\partial M}{\partial T}\right)_J = \frac{T}{\left(\frac{\partial T}{\partial S}\right)_J} \tag{3.9}$$

and is evaluated as

$$C = \frac{S\alpha\left[\beta + \left(8KJ\pi^2 + S^2\right)\left(1-K^2\right)\right]}{\alpha\beta + S^2\left(1-K^2\right)\left(S^2 + 24KJ\pi^2\right)} \tag{3.10}$$

where $\alpha = 16\pi^2 KJ + S^2$ and $\beta = \left(S^2\left(1-K^2\right)\alpha\right)^{\frac{1}{2}}$. It may be checked that the specific heat is positive ensuring local thermodynamic stability. Using eq. (3.10), we calculate the

JHEP11(2006)015

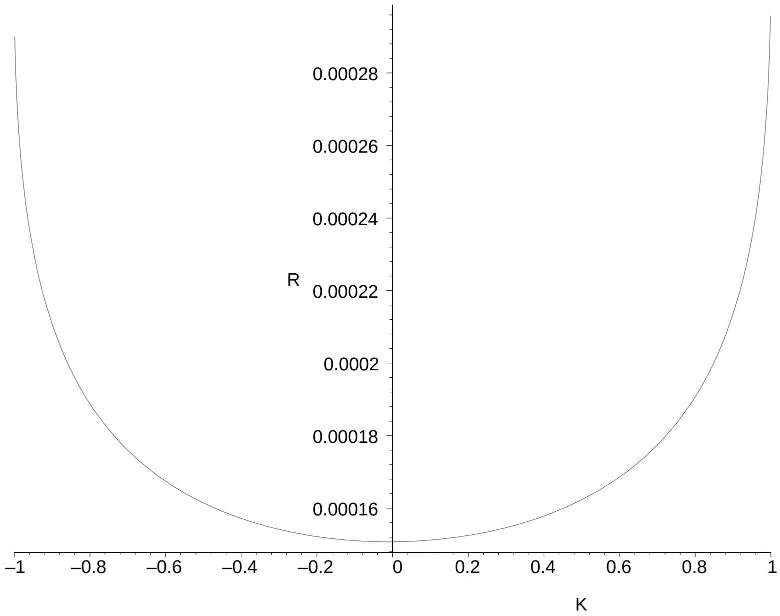

Figure 3: The Ricci scalar R of the Ruppeiner metric for the BTZ-CS black hole, as a function of the Chern-Simons coupling K, with only the logarithmic correction to the entropy being taken into account. The mass M has been set to 100 and we have set $J = 1$ to ensure that we are far from extremality.

correction to the canonical entropy including small thermal fluctuations of the statistical system and this leads to,

$$S = S_0 - \frac{1}{2}\ln CT^2 \tag{3.11}$$

where S_0 is the entropy (3.1) of the BTZ-CS in the canonical ensemble. We approximate (3.11) in the limit of large entropy, following [15]. It may be easily examined that in the limit of $S \gg J^2$ which is the stability bound, the above formula reduces to

$$S = S_0 - \frac{3}{2}\ln S_0 \tag{3.12}$$

It is interesting to note that the factor of $\frac{3}{2}$, first calculated in [16] is reproduced for the BTZ-CS black hole as well illustrating the seeming universality of this factor. This is one of the main result of this subsection.

We now calculate the Ruppeiner geometry correspoding to the modified entropy of the BTZ-CS black hole with thermal fluctuations. As in the last section, we first present the numerical result for the Ricci scalar using the leading order correction of eq. (3.12).

This is depicted in figure (3). In this analysis, we have set $M = 100$ and $J = 1$, to ensure that we are far from extremality. The Ricci scalar is positive definite in this case. The Ricci scalar diverges for $|K| = 1$. More appropriately, since we had set the cosmological constant to unity, it is not difficult to see that the bound on K from the Ruppeiner geometry is $|K| \le l$ where l is the cosmological constant. This is of course as expected, since the entropy (3.4) becomes unphysical beyond this limit.

JHEP11(2006)015

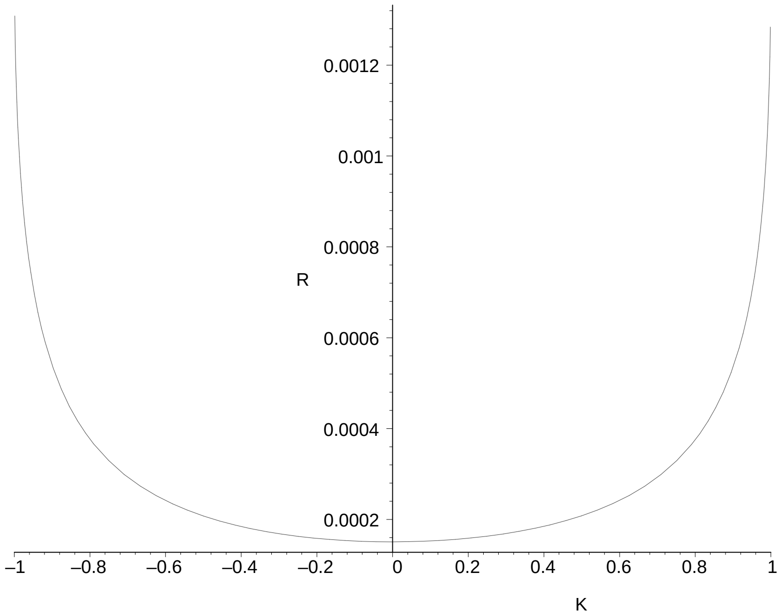

Figure 4: The Ricci scalar R of the Ruppeiner metric for the BTZ-CS black hole, as a function of the Chern-Simons coupling K. The mass M has been set to 100 and we have set $J = 1$ to ensure that we are far from extremality.

In figure (4), we present the result for the Ricci scalar of the Ruppeiner geometry taking into account the full correction of eq. (3.11). Again, as a function of K, the curvature scalar is positive definite and the graph has the same qualitative features as in figure (3).

For the sake of completeness, we have also numerically evaluated the Ricci scalar of the Ruppeiner metric for the BTZ-CS black hole as a function of the angular momentum, and studied its behaviour. The plots in this case are qualitatively the same as in figure (1) and figure (2) and we do not discuss them further.

4. Discussions and conclusions

In this article, we have mainly investigated the thermodynamic geometry of a class of BTZ black holes, both with and without the Chern-Simons term. We have shown that the Ruppeiner geometry remains flat even with the introduction of the Chern-Simons term, as it was without this term. However, introducing small thermal fluctuations in the analysis produces a non-zero Ricci scalar for the thermodynamic geometry, and we have calculated this quantity for some special cases. As a byproduct of our calculations, we have shown that the leading logarithmic correction to the canonical entropy of the BTZ-CS black hole retains the same form as for the ordinary rotating BTZ black hole thus illustrating the universality of this correction. We should mention here that the validity of this analysis depends on the local thermodynamic stability which is ensured by a positive specific heat for the BTZ and the BTZ-CS black holes. This is also generaly true for charged and rotating charged black holes. It would be intetesting to extend our analysis to other black holes and investigate the subtle interplay between the corrections due to thermal fluctutaions and α' corrections resulting from higher derivative terms. It is expected that the corresponding thermodynamic curvatures would be sensitive to these corrections. Furthermore thermodynamic geometries provide a direct way to analyse critical points of black hole phase transitions which is an area of current interest. This may have important implications for black holes in string theory and the geometry of moduli spaces. Some of these issues will be investigated in future.

A few comments are in order here. It is clear that our analysis will be similar for BTZ black holes with higher derivative corrections. As shown in [12] and [17], the form of the entropy for the BTZ-CS black hole remains the same in the presence of the higher derivative corrections, and it is the central charge of the underlying conformal field theory that is modified. Hence, we expect qualitatively similar results for the thermodynamic geometry of BTZ-CS black holes with higher derivative corrections. We have explicitly verified this. The thermodynamic geometry of BTZ black holes turn out to be insensitive to the higher derivative corrections and remains flat. A generalization of the Ruppeiner metric in terms of the Hessian of the Sen entropy function also shows that the resulting geometry is flat away from the attractor point. This is due to the fact that the effective moduli space of BTZ black holes is one dimensional with no interesting geometric structures. Currently we are exploring the extremal small black holes in N=2 supergravity and preliminary computations clearly shows that the thermodynamic geometry is sensitive to the higher derivative corrections.

As we have pointed out earlier, leading logarithmic correction to the black hole entropy arises from various sources. The black hole considered as a canonical ensemble admits such corrections to the entropy due to standard thermal fluctuations. The effect of such fluctuations may be analysed from purely thermodynamic considerations. It is to be understood that these fluctuations vanish in the thermodynamic limit of large systems where the canonical and the microcanonical entropy becomes identical. Apart from these the black hole entropy also admits logarithmic corrections due to presence of higher derivative terms to the gravitational action from the perspective of low energy effective field theories resulting from some underlying theory of quantum gravity. These higher derivative corrections are accessible to analysis through purely gravtitational considerations like Walds formula or through gravitational anomalies. It is a meaningful exercise to analyse these two corrections simultaneously and in certain cases leads to a cancellation. However corrections due to purely quantum effects must be considered separately. Lacking a viable fundmaental theory of quantum gravity these quantum corrections still need to be elucidated.

We should also remark here that as pointed out in [13] that the modification of the entropy due to the gravitational Chern-Simons term being dependent on the radius of the inner horizon seems to probe the black hole interior. This is in contrast to the higher derivative α' corrections which are only dependent on the radii of the outer horizon. This seems to indicate that contrary to the existing point of view certain degrees of freedom may be associated with the black hole interior. This may have implications for space time holography and is an important issue for future investigations.

Furthermore work is in progress to generalize the notion of thermodynamic geometries to extremal black holes in string theory, away from the attractor fixed points through the

Hessian of the corresponding *entropy functions*. Following [6] these are expected to lead to interesting insights into the nature of the moduli spaces and understanding of the full structure of the interpolating solutions away from the attractor fixed point at the horizon. It should be possible to understand the attractor mechanism as renormalization group flows in the space of thermodynamic geometries in this framework.

Acknowledgments

We are grateful to Ron-Gen Cai for helpful email correspondence. All of us would like to thank Geetanjali Sarkar for computational support and V. Subrahmanyam for discussions. GS would like to thank Sutapa Mukherji for a reference. BNT acknowldges CSIR, India, for financial support under the research grant CSIR-SRF-9/92(343)/2004-EMR-I.

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